Induction

•If we have a propositional function P(n), and we want to prove that P(n) is true for any natural number n, we do the following:

- Show that P(0) is true.
  (basis step)
- Show that if P(n) then P(n + 1) for any $n \in N$.
  (inductive step)
- Then P(n) must be true for any $n \in N$.
  (conclusion)

Induction

- **Example:**

- Show that $n < 2^n$ for all positive integers n.

- Let P(n) be the proposition "$n < 2^n$."

- 1. Show that P(1) is true.
  (basis step)

- P(1) is true, because $1 < 2^1 = 2$.

Induction

- 2. Show that if P(n) is true, then P(n + 1) is true. (inductive step)

- Assume that $n < 2^n$ is true.
- We need to show that P(n + 1) is true, i.e.
- $n + 1 < 2^{n+1}$

- We start from $n < 2^n$:
- $n + 1 < 2^n + 1 \leq 2^n + 2^n = 2^{n+1}$
- Therefore, if $n < 2^n$ then $n + 1 < 2^{n+1}$

Induction

- Then P(n) must be true for any positive integer. (conclusion)
- $n < 2^n$ is true for any positive integer.
- End of proof.

Induction

- **Another Example (“Gauss”):**
- $1 + 2 + \ldots + n = n\,(n + 1)/2$
- Show that P(0) is true.
  (basis step)
- For $n = 0$ we get $0 = 0$. True.

Induction

- Show that if P(n) then P(n + 1) for any $n \in N$. (inductive step)

- $1 + 2 + \ldots + n = n(n + 1)/2$
- $1 + 2 + \ldots + n + (n + 1) = n(n + 1)/2 + (n + 1)$
- $= (2n + 2 + n(n + 1))/2$
- $= (2n + 2 + n^2 + n)/2$
- $= (2 + 3n + n^2)/2$
- $= (n + 1)(n + 2)/2$
- $= (n + 1)((n + 1) + 1)/2$

Induction

- Then P(n) must be true for any $n \in N$. (conclusion)
- $1 + 2 + \ldots + n = n(n + 1)/2$ is true for all $n \in N$.
- End of proof.

Induction

•There is another proof technique that is very similar to the principle of mathematical induction.

•It is called **the second principle of mathematical induction (AKA strong induction)**.

•It can be used to prove that a propositional function P(n) is true for any natural number n.

Induction

- The second principle of mathematical induction:

- Show that P(0) is true.
  (basis step)
- Show that if P(0) and P(1) and … and P(n), then P(n + 1) for any $n \in N$.
  (inductive step)
- Then P(n) must be true for any $n \in N$.
  (conclusion)

Induction

- Example: Show that every integer greater than 1 can be written as the product of primes.

- Show that P(2) is true.
  (basis step)

- 2 is the product of one prime: itself.

Induction

- Show that if P(2) and P(3) and … and P(n), then P(n + 1) for any $n \in N$. (inductive step)
- Two possible cases:
- If (n + 1) is **prime**, then obviously P(n + 1) is true.
- If (n + 1) is **composite**, it can be written as the product of two integers a and b such that $2 \leq a \leq b < n + 1$.
- By the **induction hypothesis**, both a and b can be written as the product of primes.
- Therefore, $n + 1 = a \cdot b$ can be written as the product of primes.

Induction

- Then P(n) must be true for any $n \in N$.
  (conclusion)

•End of proof.

•We have shown that **every integer greater than 1** can be written as the product of primes.

If I told you once, it must be...

Recursion

Recursive Definitions

- **Recursion** is a principle closely related to mathematical induction.

- In a **recursive definition**, an object is defined in terms of itself.

- We can recursively define **sequences**, **functions** and **sets**.

Recursively Defined Sequences

- **Example:**

- The sequence $\{a_n\}$ of powers of 2 is given by
$a_n = 2^n$ for $n = 0, 1, 2, \dots$.

- The same sequence can also be defined **recursively**:

- $a_0 = 1$
- $a_{n+1} = 2a_n$ for $n = 0, 1, 2, \dots$

- Obviously, induction and recursion are similar principles.